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TECHNICAL NOTE R-3

FLARED CONE PROJECTED AREAS

Prepared By

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FLARED-CONE PROJECTED AREAS

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INTRODUCTION

This technical note presents formulas suitable for the calculation of the projected areas of arbitrary, axially symmetric surface sections for a flared-cone figure. The formulas apply to aspect angles ranging from 0 to 90°, measured from the symmetry axis. Derivations are included. These are partially based upon work contained in the Brown Engineering Company Technical Note R-18 titled "Projected Areas of Axially Symmetric Surface Sections for Several Geometric Figures", (May, 1962).

PART I PRELIMINARY DERIVATION

Figure (1) illustrates the flared-cone configuration dealt with in this report. For convenience, the locations of the bands whose projected areas are to be found will be specified in terms of the distance from the front cone apex. As indicated in figure (1), the unprimed h's are distances measured from the front cone apex. The primed h's give the distances from the rear cone apex position. Subscripts 1 and 2 refer to the front and rear band borders, respectively.

The relationship between measurements in the h and h' systems is derived with the aid of figure (2):

$$h_1 = h_1' + Z \qquad h_1' = h_1 - Z \tag{1}$$

$$Q + Z = h_{01} \tag{2}$$

$$\tan\alpha_1 = \frac{D_1}{2(Q + Z)} \qquad Z = \frac{D_1}{2\tan\alpha_1} - Q \tag{3}$$

$$\tan\alpha_2 = \frac{D_1}{2Q} \qquad Q = \frac{D_1}{2\tan\alpha_2} \tag{4}$$

Substituting the value of Q from equation (4) into equation (3) gives:

$$Z = \frac{D_1}{2\tan\alpha_1} - \frac{D_1}{2\tan\alpha_2} \tag{5}$$

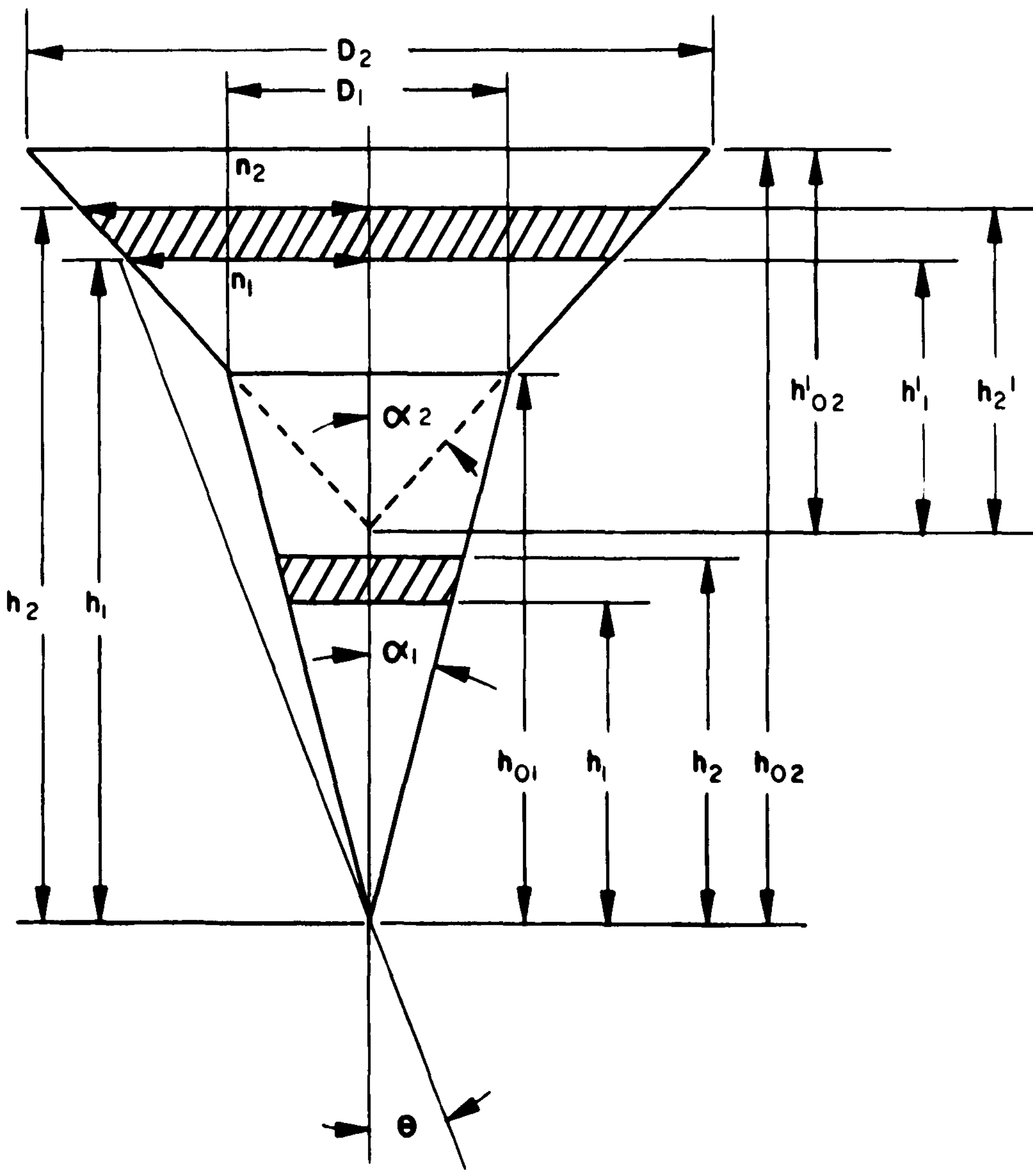

FIGURE 1

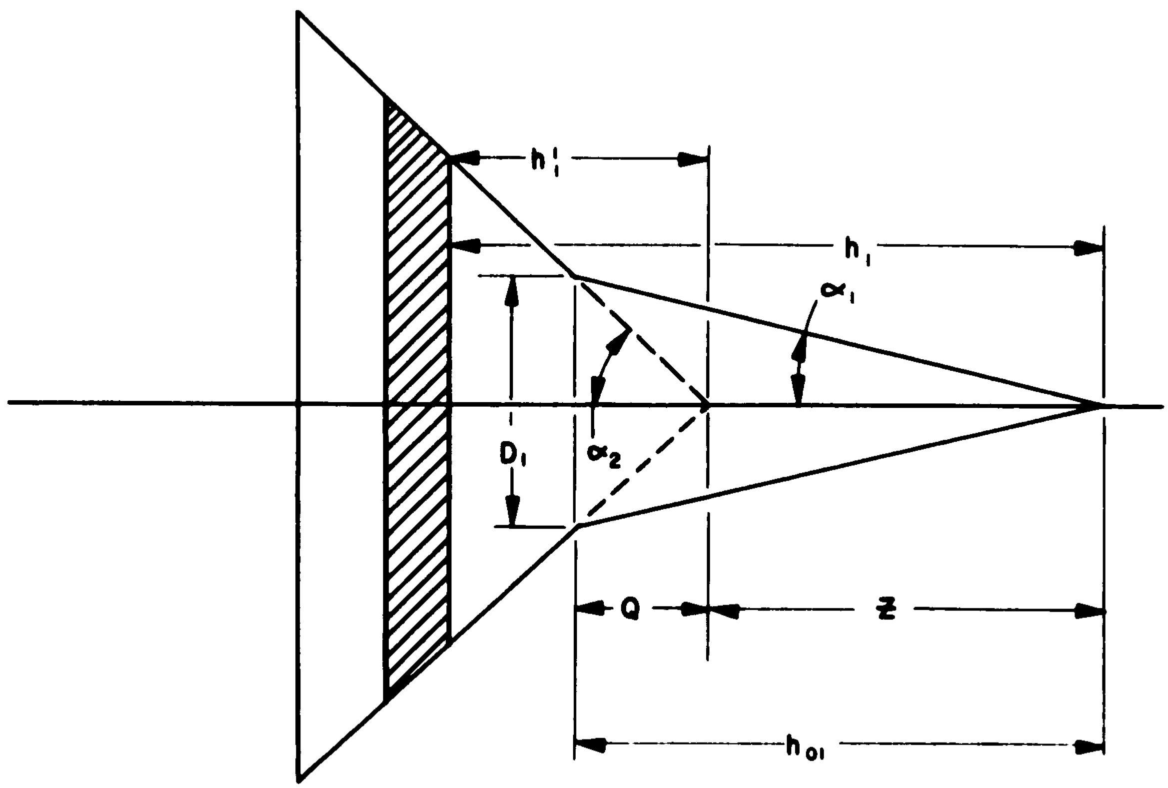

FIGURE 2

Substituting this value of Z into equation (1) gives:

$$h_1' = h_1 - \frac{D_1}{2}\left(\cot\alpha_1 - \cot\alpha_2\right) \tag{6}$$

Since $\frac{D_1}{2} = h_{01} \tan\alpha_1$, (7)

equation (6) may be rewritten:

$$h_1' = h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right) \tag{8}$$

or, dropping subscripts, we have for the general case:

$$h' = h - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right) \tag{9}$$

PART II FRONT CONE PROJECTED AREAS

Projected areas for bands lying on the front cone may be calculated with the aid of the equations derived in the May, 1962, report referred to in the introduction.

$$A = \pi \tan^2\alpha_1 \cos\theta \,(h_2^{\,2} - h_1^{\,2}) \tag{10}$$

$$0 \leqq \theta \leqq \alpha_1 \qquad\qquad h_{01} \geqq h_2 > h_1 \tag{11}$$

$$A = (h_2^{\,2} - h_1^{\,2}) \tan\alpha_1 \left(\tan\alpha_1 \cos\theta \left\{ \pi - \sin^{-1}\left[\frac{\left(\sin^2\theta - \tan^2\alpha_1 \cos^2\theta\right)^{\frac{1}{2}}}{\sin\theta} \right] \right\} + (\sin^2\theta - \tan^2\alpha_1 \cos^2\theta)^{\frac{1}{2}} \right) \tag{12}$$

$$\alpha_1 < \theta \leqq 90^\circ \qquad\qquad h_{01} \geqq h_2 > h_1 \tag{13}$$

PART III REAR CONE PROJECTED AREAS

III a) Figure 3 illustrates that for aspect angles less than or equal to α_1, the front cone does not obscure bands lying on the rear cone. Since $\alpha_2 > \alpha_1$, this case may be dealt with by the method of ellipse subtraction. Formula (10) may be applied directly.

$$A = \pi \tan^2\alpha_2 \cos\theta \left(h_2'^2 - h_1'^2\right) \tag{14}$$

or, in the unprimed system,

$$A = \pi \tan^2\alpha_2 \cos\theta \left\{ \left(h_2 - h_{01} \tan\alpha_1 \left[\cot\alpha_1 - \cot\alpha_2\right]\right)^2 - \left(h_1 - h_{01} \tan\alpha_1 \left[\cot\alpha_1 - \cot\alpha_2\right]\right)^2 \right\} \tag{15}$$

$$0 \leqq \theta \leqq \alpha_1 \qquad h_2 > h_1 \geqq h_{01} \qquad \alpha_2 > \alpha_1 \tag{16}$$

III b) For $\alpha_1 < \theta \leqq \alpha_2$, we have the case illustrated in figure (4). This case is distinguished from case III a) by the fact that in the projected view, the front cone apex is found on the rear cone projected surface. Rear cone band borders appear as non-intersecting ellipses.

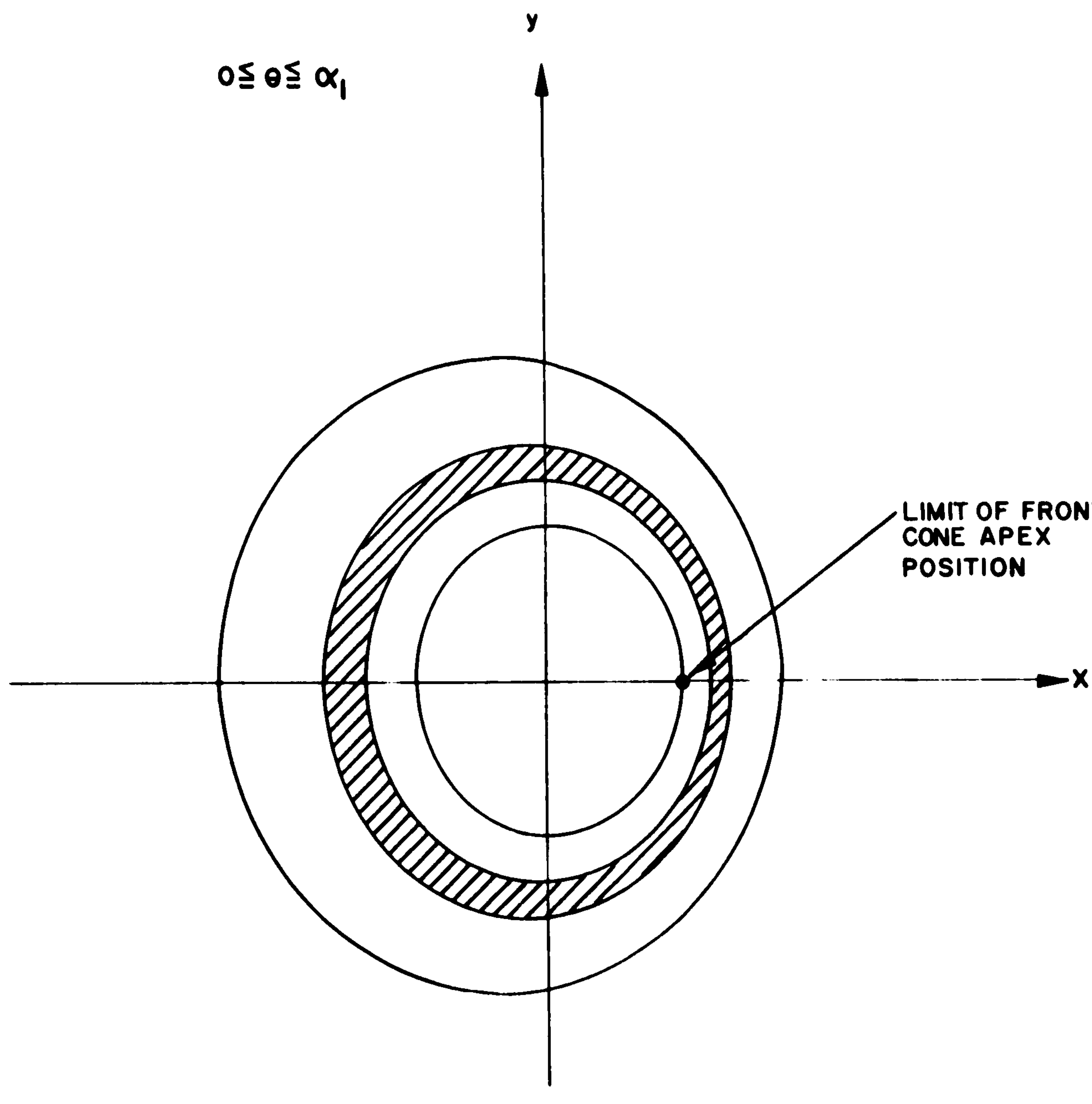

FIGURE 3

$\alpha_1 < \theta \leq \alpha_2$

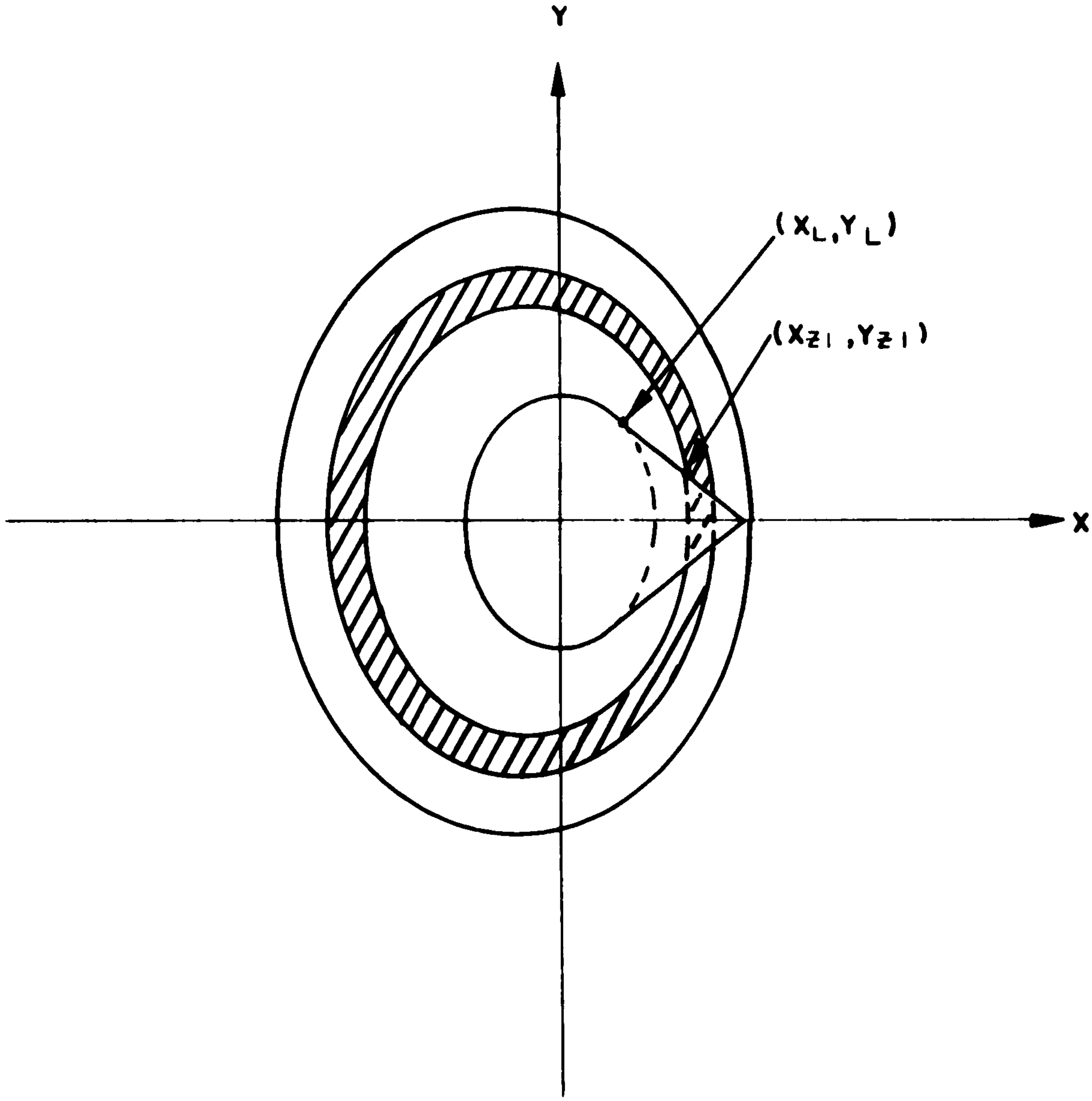

FIGURE 4

Three distinct subcases occur for this range of θ:

III b) (1) The projected apex point of the front cone may lie inside both band borders.

III b) (2) It may lie on the band.

III b) (3) It may lie outside both band borders.

Since the projected areas for rear cone bands depend upon projected apex location, the relationship between h, θ, α_1, α_2, (where h corresponds to the position of the projected apex point) is derived with the aid of figure (1).

$$\tan\theta = \frac{n_1}{h_1} \qquad \tan\alpha_2 = \frac{n_1}{h_1'} \tag{17}$$

$$h_1 \tan\theta = h_1' \tan\alpha_2 \tag{18}$$

$$\tan\theta = \frac{h_1' \tan\alpha_2}{h_1} \tag{19}$$

or, dropping subscripts on h,

$$\tan\theta = \left[h - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \left[\frac{\tan\alpha_2}{h}\right] \tag{20}$$

$$\theta = \tan^{-1}\left\{\left[h - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \left[\frac{\tan\alpha_2}{h}\right]\right\} \tag{21}$$

The analytic conditions on θ defining the three subcases listed above are:

III b) (1)

$$\theta \leqq \tan^{-1} \left\{ \left[h_1 - h_{01} \tan \alpha_1 \left(\cot \alpha_1 - \cot \alpha_2 \right) \right] \left[\frac{\tan \alpha_2}{h_1} \right] \right\} \quad (22)$$

$$\alpha_1 < \theta \leqq \alpha_2 \qquad\qquad h_2 > h_1 \geqq h_{01} \quad (23)$$

III b) (2)

$$\tan^{-1} \left\{ \left[h_1 - h_{01} \tan \alpha_1 \left(\cot \alpha_1 - \cot \alpha_2 \right) \right] \left[\frac{\tan \alpha_2}{h_1} \right] \right\} < \theta$$

$$< \tan^{-1} \left\{ \left[h_2 - h_{01} \tan \alpha_1 \left(\cot \alpha_1 - \cot \alpha_2 \right) \right] \left[\frac{\tan \alpha_2}{h_2} \right] \right\} \quad (24)$$

$$\alpha_1 < \theta \leqq \alpha_2 \qquad\qquad h_2 > h_1 \geqq h_{01} \quad (25)$$

III b) (3)

$$\tan^{-1} \left\{ \left[h_2 - h_{01} \tan \alpha_1 \left(\cot \alpha_1 - \cot \alpha_2 \right) \right] \left[\frac{\tan \alpha_2}{h_2} \right] \right\} \leqq 0 \quad (26)$$

$$\alpha_1 < \theta \leqq \alpha_2 \qquad\qquad h_2 > h_1 \geqq h_{01} \quad (27)$$

The case III b) (1) may be treated by ellipse subtraction:

$$A = \pi \tan^2\alpha_2 \cos\theta \left\{ \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right]^2 - \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right]^2 \right\} \tag{28}$$

$$\theta \leqq \tan^{-1} \left\{ \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \left[\frac{\tan\alpha_2}{h_1} \right] \right\} \tag{29}$$

$$\alpha_1 < \theta \leqq \alpha_2 \qquad\qquad h_2 > h_1 \geqq h_{01} \tag{30}$$

The case III b) (2) is dealt with by calculating the projected area of the band by ellipse subtraction and then subtracting from this result the area of the apex falling inside the band (figure 4), thus:

$$A = \pi \tan^2\alpha_2 \cos\theta \left\{ \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right]^2 - \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right]^2 \right\} - A' \tag{31}$$

$$\tan^{-1} \left\{ \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \left[\frac{\tan\alpha_2}{h_1} \right] \right\} < 0 < \tan^{-1} \left\{ \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \left[\frac{\tan\alpha_2}{h_2} \right] \right\} \tag{32}$$

$$\alpha_1 < \theta \leqq \alpha_2 \qquad\qquad h_2 > h_1 \geqq h_{01} \tag{33}$$

where A' is the apex area falling within the projected band.

To calculate A', we write the equation of the projected apex line and the equation of the inner-band ellipse and integrate (figure 4) between the line and the ellipse:

$$A' = 2 \int_0^{y_{z1}} (X_{fcl} - X_{eh_1})\, dy \qquad (34)$$

where the subscript fcl denotes front cone line and eh_1 denotes the ellipse corresponding to h_1.

With the origin of coordinates at the center of front cone base ellipse

$$X_{fcl} = \frac{1}{m_{fcl}} y + C_{fcl} \text{(from equation (14), May 1962 Report)} \qquad (35)$$

where m_{fcl} and C_{fcl} refer to the front cone line projection.

In the same coordinate system, the equation of the ellipse is:

$$\frac{(X + W)^2}{a'^2} + \frac{y^2}{b'^2} = 1 \qquad (36)$$

or,

$$X_{eh_1} = \frac{a'}{b'} (b'^2 - y^2)^{\frac{1}{2}} - W \qquad (37)$$

With these substitutions for X_{fc1} and X_{eh_1}, the integral A' becomes:

$$A' = 2 \int_0^{y_{z1}} \left\{ \left[\frac{1}{m_{fc1}} y + C_{fc1} \right] - \left[\frac{a'}{b'} (b'^2 - y^2)^{\frac{1}{2}} - W \right] \right\} dy \quad (38)$$

$$= \frac{2}{m_{fc1}} \int_0^{y_{z1}} y\,dy + 2C_{fc1} \int_0^{y_{z1}} dy - 2\frac{a'}{b'} \int_0^{y_{z1}} (b'^2 - y^2)^{\frac{1}{2}} dy$$

$$+ 2W \int_0^{y_{z1}} dy \quad (39)$$

which, when integrated, gives:

$$A' = \frac{y_{z1}^2}{m_{fc1}} + 2C_{fc1}\, y_{z1} - \frac{a'}{b'} \left[y_{z1} (b'^2 - y_{z1}^2)^{\frac{1}{2}} + b'^2 \sin^{-1} \left(\frac{y_{z1}}{b'} \right) \right]$$

$$+ 2Wy_{z1} \quad (40)$$

The remaining part of the solution consists of expressing C_{fc1}, a', b', W, m_{fc1} and y_{z1} in terms of α_1, α_2, θ, h_1, and h_{01}.

We have:

$$C_{fc1} = h_{01} \sin \theta \quad (41)$$

$$a' = h_1' \tan \alpha_2 \cos \theta$$

$$= \left[h_1 - h_{01} \tan \alpha_1 \left(\cot \alpha_1 - \cot \alpha_2 \right) \right] \tan \alpha_2 \cos \theta \quad (42)$$

$$b' = h_1' \tan \alpha_2 = \left[h_1 - h_{01} \tan \alpha_1 (\cot \alpha_1 - \cot \alpha_2) \right] \tan \alpha_2 \tag{43}$$

$$W = (h_1 - h_{01}) \sin \theta \tag{44}$$

$$m_{fc1} = -\left(\frac{b^2}{C^2 - a^2} \right)^{\frac{1}{2}} = - \frac{\tan \alpha_1}{(\sin^2 \theta - \tan^2 \alpha_1 \cos^2 \theta)^{\frac{1}{2}}} \tag{45}$$

Since y_{z1} is the y coordinate of the point of intersection of the projected ellipse corresponding to h_1 and the projected cone line corresponding to the front cone with altitudes h_{01}, we solve the equation of the ellipse and cone line simultaneously to obtain y_{z1}.

$$X_{z1} = \frac{a'}{b'} (b'^2 - y_{z1}^2)^{\frac{1}{2}} - W \tag{46}$$

$$X_{z1} = \frac{1}{m_{fc1}} y_{z1} + C_{fc1} \tag{47}$$

$$\frac{1}{m_{fc1}} y_{z1} + C_{fc1} = \frac{a'}{b'} (b'^2 - y_{z1}^2)^{\frac{1}{2}} - W \tag{48}$$

$$\frac{1}{m_{fc1}} y_{z1} + C_{fc1} + W = \frac{a'}{b'} (b'^2 - y_{z1}^2)^{\frac{1}{2}} \tag{49}$$

Squaring both sides of equation (49) gives

$$\frac{1}{m^2_{fc1}} y_{z1}^2 + \frac{2}{m_{fc1}} (C_{fc1} + W) y_{z1} + (C_{fc1} + W)^2 = \frac{a'^2}{b'^2} (b'^2 - y_{z1}^2)$$

$$= a'^2 - \frac{a'^2}{b'^2} y_{z1}^2 \tag{50}$$

or, temporarily dropping subscripts for economy in writing,

$$(\frac{1}{m^2} + \frac{a'^2}{b'^2}) y_{z1}^2 + \frac{2}{m} (C + W) y_{z1} + \left[(C + W)^2 - a'^2 \right] = 0 \tag{51}$$

The solution of this quadratic form is:

$$y_{z1} = \frac{-B \pm (B^2 - 4AG)^{\frac{1}{2}}}{2A} \tag{52}$$

where:

$$A = (\frac{1}{m^2} + \frac{a'^2}{b'^2}) \tag{53}$$

$$B = \frac{2}{m}(C + W) \tag{54}$$

$$G = (C + W)^2 - a'^2 \tag{55}$$

$$y_{z1} = \left[- \frac{2}{m}(C+W) \pm \left\{ \left[\frac{2}{m}(C+W) \right]^2 - 4\left(\frac{1}{m^2} + \frac{a'^2}{b'^2} \right) \left((C+W)^2 - a'^2 \right) \right\}^{\frac{1}{2}} \right] \div 2\left(\frac{1}{m^2} + \frac{a'^2}{b'^2} \right) \tag{56}$$

$$y_{z1} = \frac{ - \frac{(C_{fc1} + W)}{m_{fc1}} - \left[\frac{a'^2}{m_{fc1}^2} - \frac{a'^2}{b'^2}(C_{fc1} + W)^2 + \frac{a'^4}{b'^2} \right]^{\frac{1}{2}} }{ \left(\frac{1}{m_{fc1}^2} + \frac{a'^2}{b'^2} \right) } \tag{57}$$

where the negative sign before the radical has been chosen to yield the smaller positive solution (see figure 4).

The case III b) (2) may be summarized:

$$A = \pi \tan^2 \alpha_2 \cos\theta \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right]^2 - \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right]^2 - \left\{ \frac{y_{z1}^2}{m_{fc1}} + 2C_{fc1} y_{z1} - \frac{a'}{b'} \left[y_{z1} \left(b'^2 - y_{z1}^2 \right)^{\frac{1}{2}} + b'^2 \sin^{-1} \frac{y_{z1}}{b'} \right] + 2W y_{z1} \right\} \tag{58}$$

$$\tan^{-1} \left\{ \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \left[\frac{\tan\alpha_2}{h_1} \right] \right\} < \theta < \tan^{-1} \left\{ \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \left[\frac{\tan\alpha_2}{h_2} \right] \right\} \tag{59}$$

$$\alpha_1 < \theta \leq \alpha_2 \qquad\qquad h_2 > h_1 \geq h_{01} \tag{60}$$

where

$$a' = \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \left[\tan\alpha_2 \cos\theta\right] \tag{61}$$

$$b' = \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \tan\alpha_2 \tag{62}$$

$$C_{fc1} = h_{01} \sin\theta \tag{63}$$

$$W = (h_1 - h_{01}) \sin\theta \tag{64}$$

$$m_{fc1} = - \frac{\tan\alpha_1}{(\sin^2\theta - \tan^2\alpha_1 \cos^2\theta)^{\frac{1}{2}}} \tag{65}$$

$$y_{z1} = \frac{-\dfrac{(C_{fc1} + W)}{m_{fc1}} - \left[\dfrac{a'^2}{m_{fc1}^2} - \dfrac{a'^2}{b'^2}(C_{fc1} + W)^2 + \dfrac{a'^4}{b'^2}\right]^{\frac{1}{2}}}{\left(\dfrac{1}{m_{fc1}^2} + \dfrac{a'^2}{b'^2}\right)} \tag{66}$$

In case III b) (3), illustrated in figure (5), the front cone apex point falls outside both back cone borders. The solution for this case may be obtained directly from the solution for case III b) (2) by the use of appropriate subscripts.

$\alpha_1 < \theta \leq \alpha_2$

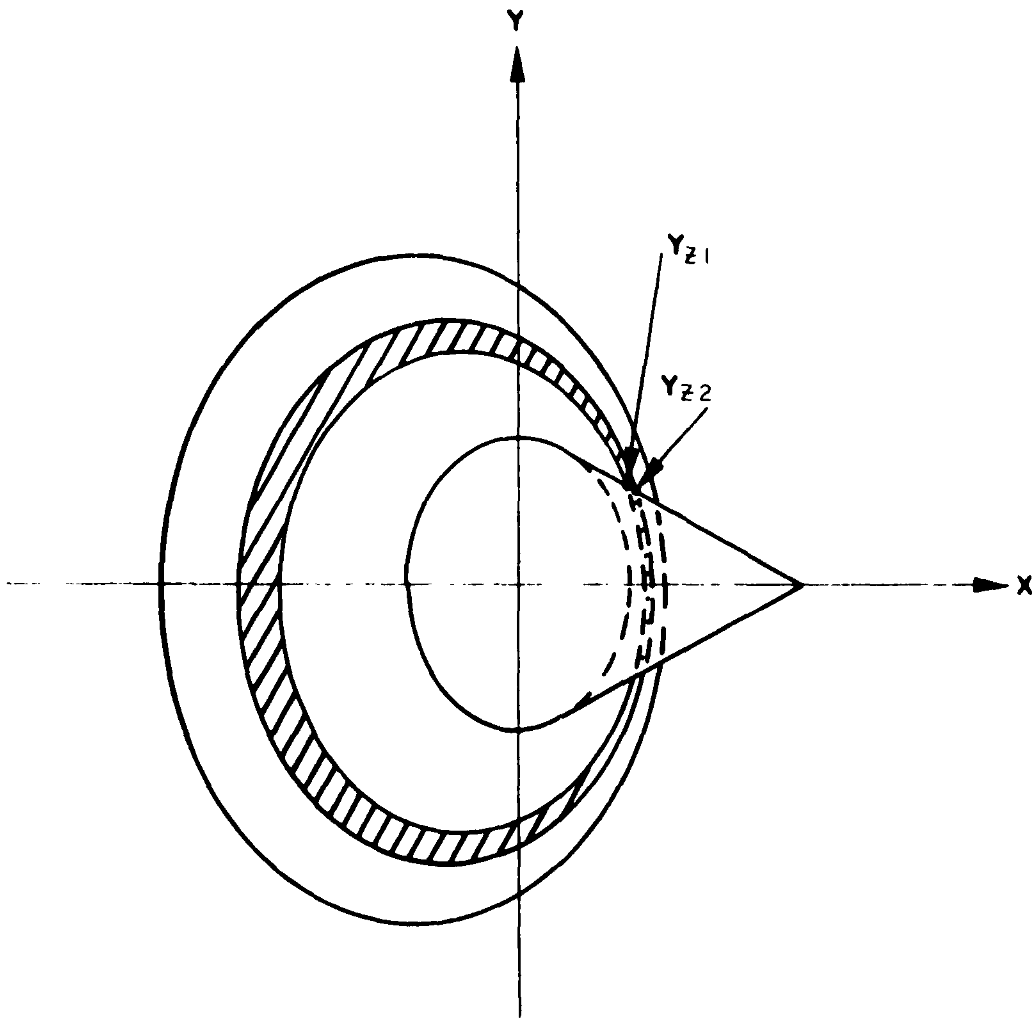

FIGURE 5

$$A = \pi \tan^2\alpha_2 \cos\theta \left\{ \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right]^2 - \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right]^2 \right\}$$

$$- \left(\frac{y_{z1}}{m_{fc1}} + 2C_{fc1} y_{z1} - \frac{a_1'}{b_1'} \left[y_{z1} \left(b_1'^2 - y_{z1}^2 \right)^{\frac{1}{2}} + b_1'^2 \sin^{-1} \frac{y_{z1}}{b_1'} \right] + 2W_1 y_{z1} - \left\{ \frac{y_{z1}}{m_{fc1}} + 2C_{fc1} y_{z2} \right.\right.$$

$$\left.\left. - \frac{a_2'}{b_2'} \left[y_{z2} \left(b_2'^2 - y_{z2}^2 \right)^{\frac{1}{2}} + b_2'^2 \sin^{-1} \frac{y_{z2}}{b_2'} \right] + 2W_2 y_{z2} \right\} \right) \quad (67)$$

$$\tan^{-1} \left\{ \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \left[\frac{\tan\alpha_2}{h_2} \right] \right\} \leq \theta \quad (68)$$

$$\alpha_1 < \theta \leq \alpha_2 \qquad h_2 > h_1 \geq h_{01} \quad (69)$$

where

$$a_1' = \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \left[\tan\alpha_2 \cos\theta \right] \quad (70)$$

$$b_1' = \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \left[\tan\alpha_2 \right] \quad (71)$$

$$a_2' = \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \left[\tan\alpha_2 \cos\theta \right] \quad (72)$$

$$b_2' = \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \left[\tan\alpha_2 \right] \quad (73)$$

$$C_{fc1} = h_{01} \sin\theta \tag{74}$$

$$W_1 = (h_1 - h_{01}) \sin\theta \tag{75}$$

$$W_2 = (h_2 - h_{01}) \sin\theta \tag{76}$$

$$m_{fc1} = - \frac{\tan\alpha_1}{(\sin^2\theta - \tan^2\alpha_1 \cos^2\theta)^{\frac{1}{2}}} \tag{77}$$

$$y_{z1} = \frac{-\left(\dfrac{C_{fc1} + W_1}{m_{fc1}}\right) - \left[\dfrac{a_1'^2}{m_{fc1}^2} - \dfrac{a_1'^2}{b_1'^2}(C_{fc1} + W_1)^2 + \dfrac{a_1'^4}{b_1'^2}\right]^{\frac{1}{2}}}{\left(\dfrac{1}{m_{fc1}^2} + \dfrac{a_1'^2}{b_1'^2}\right)} \tag{78}$$

$$y_{z2} = \frac{-\left(\dfrac{C_{fc1} + W_2}{m_{fc1}}\right) - \left[\dfrac{a_2'^2}{m_{fc1}^2} - \dfrac{a_2'^2}{b_2'^2}(C_{fc1} + W_2)^2 + \dfrac{a_2'^4}{b_2'^2}\right]^{\frac{1}{2}}}{\left(\dfrac{1}{m_{fc1}^2} + \dfrac{a_2'^2}{b_2'^2}\right)} \tag{79}$$

III c) For $\alpha_2 < \theta < 90°$, the equations listed in the May, 1962, report apply. However, special consideration must be given to bands that are partially obscured by the straight line portion of the projected front cone.

Figure (6) illustrates the three possibilities for this range of aspect angle.

$\alpha_2 < \theta \leqq 90°$

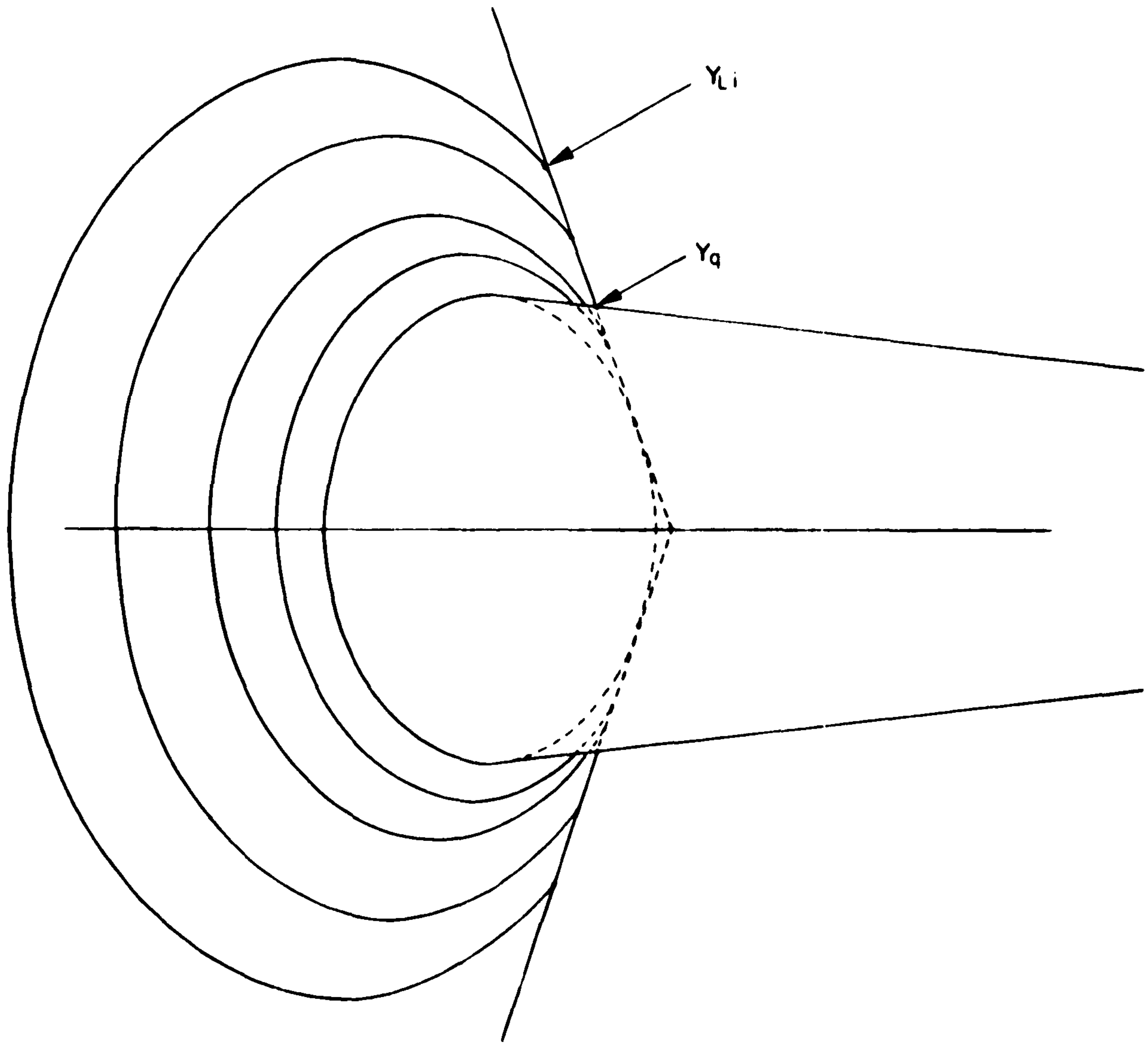

FIGURE 6

III c) (1) Band not obscured by front cone.

III c) (2) One band border falls behind front cone.

III c) (3) Both band borders fall behind front cone.

These three cases may be analytically defined by the following relationships:

III c) (1) $y_{L2} > y_{L1} \geqq y_q$ (figure 7) (80)

III c) (2) $y_{L2} \geqq y_q > y_{L1}$ (figure 8) (81)

III c) (3) $y_q > y_{L2} > y_{L1}$ (figure 9) (82)

where y_{L2} and y_{L1} are the y coordinates of the points of tangency of the rear and front band borders respectively with the straight portion of the rear cone projection, and y_q is the y coordinate of the point of intersection of the straight line cone projections.

Case III c) (1), figure 7, may be treated by formula 33, May, 1962, report. The condition

$$y_{L2} > y_{L1} \geqq y_q \tag{83}$$

may be expressed in the form (from equation 31, May, 1962, report)

$$\frac{b_2'}{C_2'} \left(C_2'^2 - a_2'^2\right)^{\frac{1}{2}} > \frac{b_1'}{C_1'} \left(C_1'^2 - a_1'^2\right)^{\frac{1}{2}} \geqq y_q \tag{84}$$

y_q is obtained by writing the equation of the two straight lines in the same coordinate system and solving simultaneously. Taking as the

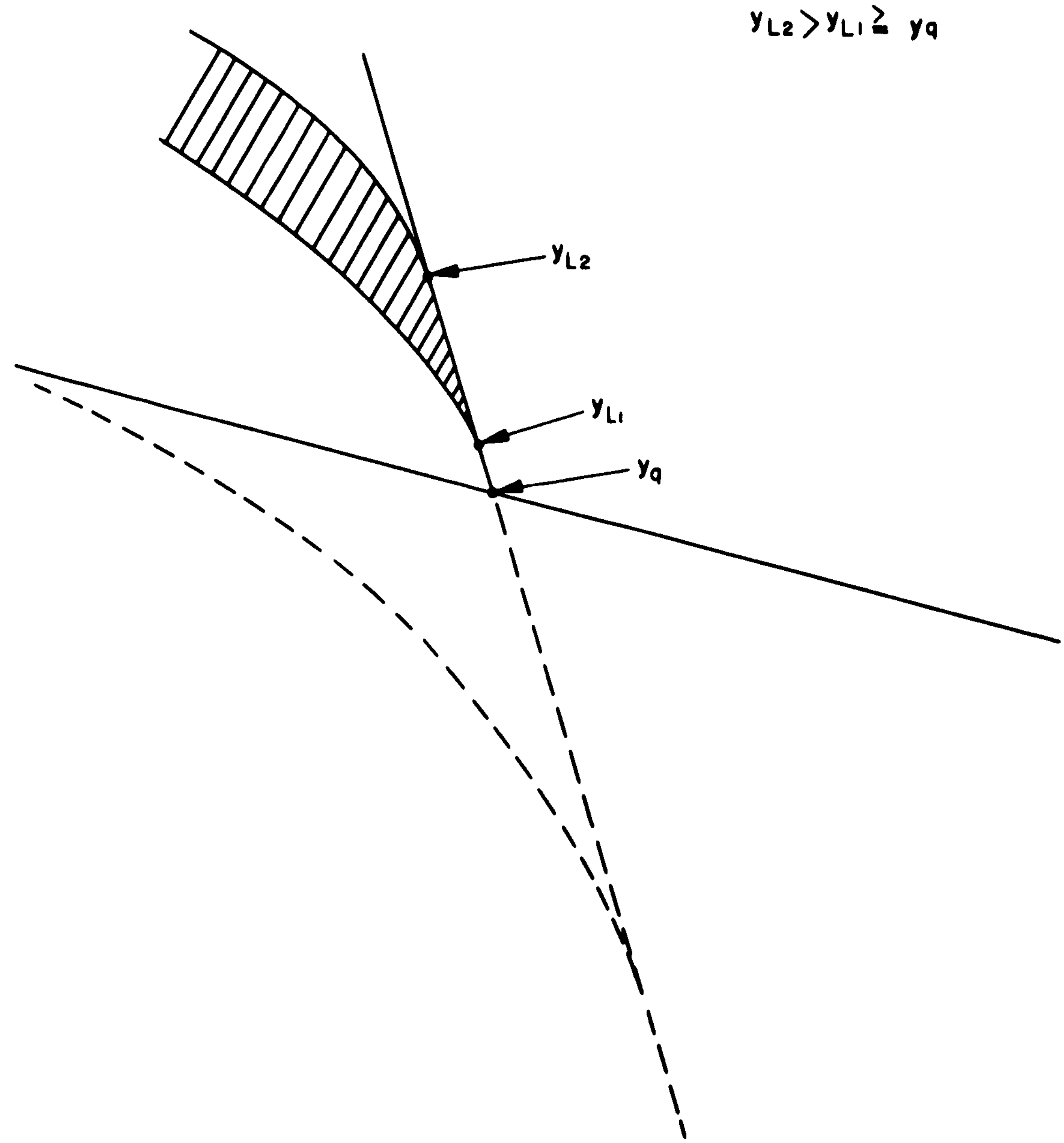

FIGURE 7

origin of coordinates the center of the front cone base ellipse, we have for the front cone line

$$X_q = \frac{1}{m_{fcl}} y_q + C_{fcl} \tag{85}$$

where the subscript fcl refers to the front cone line.

For the rear cone line equation we have:

$$X_q = \frac{1}{m_{rcl}} y_q + C_{rcl} \tag{86}$$

or

$$X_q = \frac{1}{m_{rcl}} y_q + Q \sin \theta \tag{87}$$

Equating equations (85) and (87)

$$\frac{1}{m_{fcl}} y_q + C_{fcl} = \frac{1}{m_{rcl}} y_q + Q \sin \theta \tag{88}$$

$$y_q = \frac{Q \sin \theta - C_{fcl}}{\dfrac{1}{m_{fcl}} - \dfrac{1}{m_{rcl}}} \tag{89}$$

where

$$Q = \frac{\tan \alpha_1}{\tan \alpha_2} (h_{01}) \qquad \text{(figure 2)} \tag{90}$$

Summarizing case III c) (1), we have

$$A = (h_2'^2 - h_1'^2)\tan\alpha_2 \left(\tan\alpha_2 \cos\theta \left\{\pi - \sin^{-1}\left[\frac{(\sin^2\theta - \tan^2\alpha_2\cos^2\theta)^{\frac{1}{2}}}{\sin\theta}\right]\right\} + (\sin^2\theta - \tan^2\alpha_2\cos^2\theta)^{\frac{1}{2}}\right) \qquad (91)$$

$$\alpha_2 < \theta \leq 90^\circ \qquad h_2 > h_1 \geq h_{01} \qquad (92)$$

$$\frac{b_2'}{C_2'}(C_2'^2 - a_2'^2)^{\frac{1}{2}} > \frac{b_1'}{C_1'}(C_1'^2 - a_1'^2)^{\frac{1}{2}} \geq \frac{Q\sin\theta - C_{fc1}}{\frac{1}{m_{fc1}} - \frac{1}{m_{rc1}}} \qquad (93)$$

where

$$h_1' = h_1 - h_{01}\tan\alpha_1\left(\cot\alpha_1 - \cot\alpha_2\right) \qquad (94)$$

$$h_2' = h_2 - h_{01}\tan\alpha_1\left(\cot\alpha_1 - \cot\alpha_3\right) \qquad (95)$$

$$a_1' = \left[h_1 - h_{01}\tan\alpha_1\left(\cot\alpha_1 - \cot\alpha_2\right)\right]\left[\tan\alpha_2\cos\theta\right] \qquad (96)$$

$$b_1' \quad \left[h_1 - h_{01}\tan\alpha_1\left(\cot\alpha_1 - \cot\alpha_2\right)\right]\left[\tan\alpha_2\right] \qquad (97)$$

$$a_2' = \left[h_2 - h_{01}\tan\alpha_1\left(\cot\alpha_1 - \cot\alpha_2\right)\right]\left[\tan\alpha_2\cos\theta\right] \qquad (98)$$

$$b_2' = \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \left[\tan\alpha_2\right] \tag{99}$$

$$C_1' = \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \sin\theta \tag{100}$$

$$C_2' = \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \sin\theta \tag{101}$$

$$C_{fc1} = h_{01} \sin\theta \tag{102}$$

$$m_{fc1} = - \frac{\tan\alpha_1}{(\sin^2\theta - \tan^2\alpha_1 \cos^2\theta)^{\frac{1}{2}}} \tag{103}$$

$$m_{rc1} = - \frac{\tan\alpha_2}{(\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}}} \tag{104}$$

$$Q = \frac{\tan\alpha_1}{\tan\alpha_2} (h_{01}) \tag{105}$$

Case III c) (2), figure 8, may be dealt with by subtracting from the projected band area given by equation (91) the area obscured by the front cone, given by

$$A' = 2 \left[\int_{y_{L1}}^{y_q} (X_{rc1} - X_{eh_1})\, dy + \int_{y_q}^{y_{z1}} (X_{fc1} - X_{eh_1})\, dy \right] \tag{106}$$

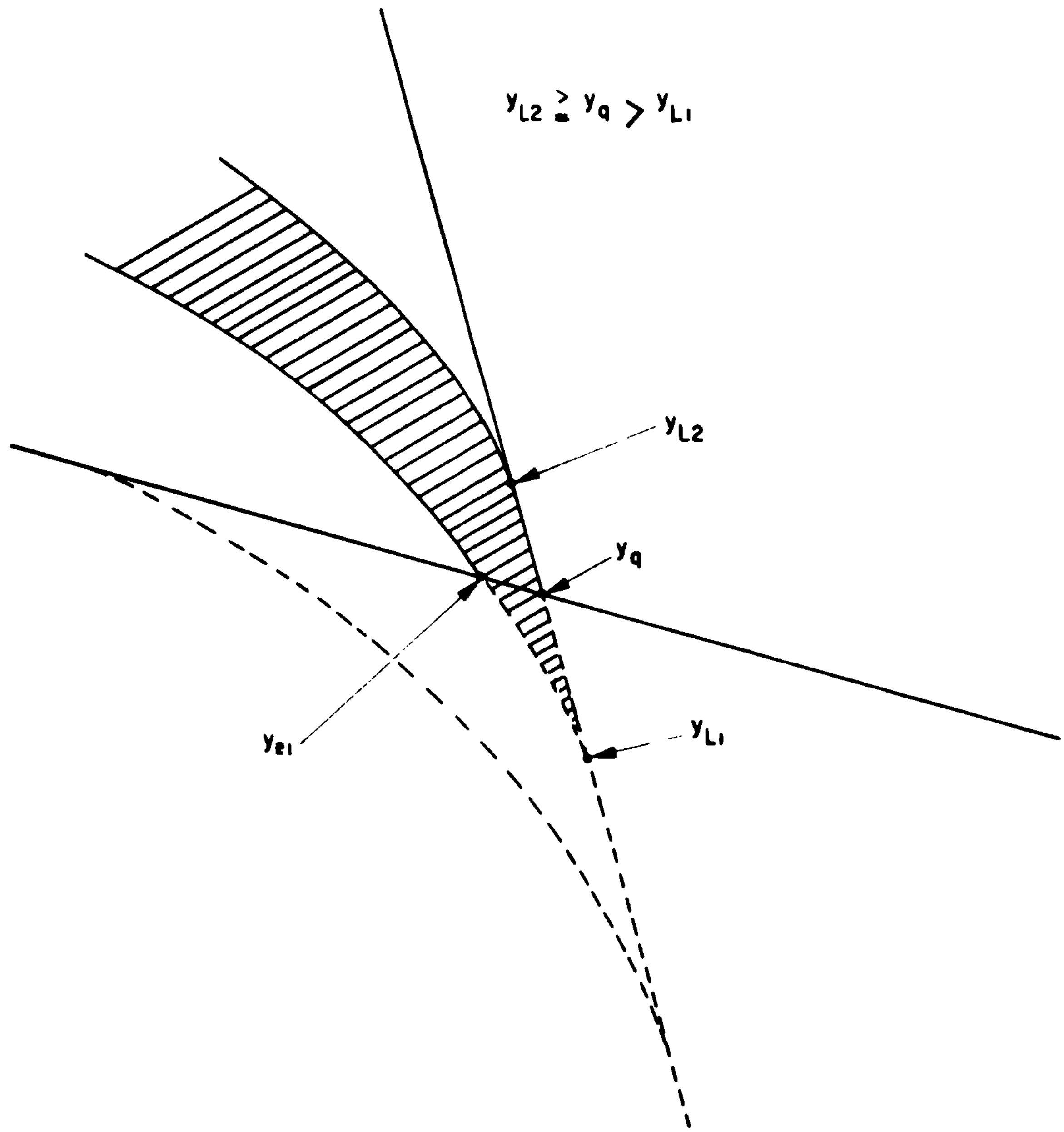

FIGURE 8

where

rcl and fcl refer to the rear cone line and the front cone line, respectively.

eh_1 denotes the projected ellipse corresponding to h_1.

y_{z1} is the y coordinate of the point of intersection of the front cone line projection and the projected ellipse corresponding to h_1.

y_{L1}, y_{L2}, y_q, and y_{z1} have been defined by equations (84), (89), and (57).

To evaluate the integral, we choose as the origin of coordinates the center of the ellipse corresponding to h_1. In this system we have:

$$X_{eh_1} = \frac{a_1'}{b_1'}(b_1'^2 - y_1'^2)^{\frac{1}{2}} \tag{107}$$

$$X_{fcl} = \frac{1}{m_{fcl}} y + C_{fcl} + P \tag{108}$$

$$X_{fcl} = \frac{1}{m_{fcl}} y + h_1 \sin \theta \tag{109}$$

To obtain the equation of the rear cone line projection in the eh_1 coordinate system, we write:

$$X_{rcl} = \frac{1}{m_{rcl}} y + C_{rcl} - V \tag{110}$$

where $C_{rcl} = h_{02}' \sin\theta$ (111)

and $V = (h_{02} - h_1) \sin\theta$ (112)

so that

$$C_{rcl} - V = h_1' \sin\theta \tag{113}$$

giving

$$X_{rcl} = \frac{1}{m_{rcl}} y + h_1' \sin\theta \tag{114}$$

We may now evaluate the integral for A'.

$$A' = 2\left[\int_{y_{L1}}^{y_q} (X_{rcl} - X_{eh_1})\,dy + \int_{y_q}^{y_{z1}} (X_{fcl} - X_{eh_1})\,dy\right] \tag{115}$$

$$= 2\left\{\int_{y_{L1}}^{y_q}\left[\frac{1}{m_{rcl}} y + h_1' \sin\theta - \frac{a_1'}{b_1'}(b_1'^2 - y^2)^{\frac{1}{2}}\right]dy\right.$$

$$\left. + \int_{y_q}^{y_{z1}}\left[\frac{1}{m_{fcl}} y + h_1 \sin\theta - \frac{a_1'}{b_1'}(b_1'^2 - y^2)^{\frac{1}{2}}\right]dy\right\} \tag{116}$$

$$A' = 2\left\{\frac{1}{m_{rc1}}\,\frac{y^2}{2}\Bigg|_{y_{L1}}^{y_q} + (h_1'\sin\theta)\,y\Bigg|_{y_{L1}}^{y_q}\right.$$

$$- \frac{a_1'}{b_1'}\int_{y_{L1}}^{y_q}(b_1'^2 - y^2)^{\frac{1}{2}}\,dy + \frac{1}{m_{fc1}}\,\frac{y^2}{2}\Bigg|_{y_q}^{y_{z1}} + (h_1\sin\theta)\,y\Bigg|_{y_q}^{y_{z1}}$$

$$\left. - \frac{a_1'}{b_1'}\int_{y_q}^{y_{z1}}(b_1'^2 - y^2)^{\frac{1}{2}}\,dy\right\} \tag{117}$$

$$A' = \frac{y_q^{\,2} - y_{L1}^{\,2}}{m_{rc1}} + 2(h_1'\sin\theta)\,(y_q - y_{L1})$$

$$- \left[\frac{a_1'}{b_1}\right]\left\{\left[y_q\,(b_1'^2 - y_q^{\,2})^{\frac{1}{2}} + b_1'^2\sin^{-1}\left(\frac{y_q}{b_1'}\right)\right]\right.$$

$$\left. - \left[y_{L1}\,(b_1'^2 - y_{L1}^{\,2})^{\frac{1}{2}} + b_1'^2\sin^{-1}\left(\frac{y_{L1}}{b_1'}\right)\right]\right\}$$

$$+ \frac{y_{z1}^{\,2} - y_q^{\,2}}{m_{fc1}} + 2\,(h_1\sin\theta)\,(y_{z1} - y_q)$$

$$- \left[\frac{a_1'}{b_1'}\right]\left\{\left[y_{z1}\,(b_1'^2 - y_{z1}^{\,2})^{\frac{1}{2}} + b_1'^2\sin^{-1}\left(\frac{y_{z1}}{b_1'}\right)\right]\right.$$

$$\left. - \left[y_q\,(b_1'^2 - y_q^{\,2})^{\frac{1}{2}} + b_1'^2\sin^{-1}\left(\frac{y_q}{b_1'}\right)\right]\right\} \tag{118}$$

or

$$A' = \frac{y_q^2 - y_{L1}^2}{m_{rc1}} + 2\,(h_1' \sin\theta)(y_q - y_{L1})$$

$$+ \left[\frac{a_1'}{b_1'}\right] \left\{ \left[y_{L1}(b_1'^2 - y_{L1}^2)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{L1}}{b_1'}\right) \right] \right.$$

$$\left. - \left[y_{z1}\,(b_1'^2 - y_{z1}^2)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{z1}}{b_1'}\right) \right] \right\}$$

$$+ \frac{y_{z1}^2 - y_q^2}{m_{fc1}} + 2\,(h_1 \sin\theta)(y_{z1} - y_q) \qquad (119)$$

Summarizing case III c) (2), we have

$$A = (h_2'^2 - h_1'^2) \tan\alpha_2 \left(\tan\alpha_2 \cos\theta \right.$$

$$\left\{ \pi - \sin^{-1}\left[\frac{(\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}}}{\sin\theta} \right] \right\}$$

$$\left. + (\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}} \right) - \left(\frac{y_q^2 - y_{L1}^2}{m_{rc1}} \right.$$

$$+ (2h_1' \sin\theta)(y_q - y_{L1}) + \left(\frac{a_1'}{b_1'}\right) \left\{ \left[y_{L1}\,(b_1'^2 - y_{L1}^2)^{\frac{1}{2}} \right.\right.$$

$$\left.\left. + b_1'^2 \sin^{-1}\left(\frac{y_{L1}}{b_1'}\right) \right] - \left[y_{z1}\,(b_1'^2 - y_{z1})^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{z1}}{b_1'}\right) \right] \right\}$$

$$\left. + \frac{y_{z1}^2 - y_q^2}{m_{fc1}} + 2\,(h_1 \sin\theta)(y_{z1} - y_q) \right) \qquad (120)$$

$$\alpha_2 < \theta < 90^{\circ} \qquad h_2 > h_1 \geq h_{01} \tag{121}$$

$$\frac{b_2'}{C_2'}\left(C_2'^2 - a_2'^2\right)^{\frac{1}{2}} \geq \frac{Q \sin\theta - C_{fc1}}{\dfrac{1}{m_{fc1}} - \dfrac{1}{m_{rc1}}} > \frac{b_1'}{C_1'}\left(C_1'^2 - a_1'^2\right)^{\frac{1}{2}} \tag{122}$$

where

$$h_1' = h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right) \tag{123}$$

$$h_2' = h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 \quad \cot\alpha_2\right) \tag{124}$$

$$a_1' = \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \left[\tan\alpha_2 \cos\theta\right] \tag{125}$$

$$b_1' = \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \left[\tan\alpha_2\right] \tag{126}$$

$$a_2' = \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \left[\tan\alpha_2 \cos\theta\right] \tag{127}$$

$$b_2' = \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \left[\tan\alpha_2\right] \tag{128}$$

$$C_1' = \left[h_1 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \sin\theta \tag{129}$$

$$C_2' = \left[h_2 - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2\right)\right] \sin\theta \tag{130}$$

$$C_{fc1} = h_{01} \sin\theta \tag{131}$$

$$m_{fc1} = \frac{-\tan\alpha_1}{\left(\sin^2\theta - \tan^2\alpha_1 \cos^2\theta\right)^{\frac{1}{2}}} \tag{132}$$

$$m_{rc1} = - \frac{\tan\alpha_2}{(\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}}} \tag{133}$$

$$Q = \frac{\tan\alpha_1}{\tan\alpha_2} (h_{01}) \tag{134}$$

$$y_{L1} = \frac{b_1'}{C_1'} (C_1'^2 - a_1'^2)^{\frac{1}{2}} \tag{135}$$

$$y_q = \frac{Q \sin\theta - C_{fc1}}{\frac{1}{m_{fc1}} - \frac{1}{m_{rc1}}} \tag{136}$$

$$y_{z1} = \frac{-\left(\frac{h_1 \sin\theta}{m_{fc1}}\right) - \left[\frac{a_1'^2}{m_{fc1}^2} - \frac{a_1'^2}{b_1'^2}(h_1 \sin\theta)^2 + \frac{a_1'^4}{b_1'^2}\right]^{\frac{1}{2}}}{\left(\frac{1}{m_{fc1}^2} + \frac{a_1'^2}{b_1'^2}\right)} \tag{137}$$

Case III c) (3), figure 9, which is defined by the condition

$$y_q > y_{L2} > y_{L1} \tag{138}$$

may be treated by subtracting from the projected band area given by equation (91) the area obscured by the front cone, given by

$$A' = 2\left[\int_{y_{L1}}^{y_{L2}} (X_{rc1} - X_{eh_1})\,dy + \int_{y_{L2}}^{y_{z2}} (X_{eh_2} - X_{eh_1})\,dy + \int_{y_{z2}}^{y_{z1}} (X_{fc1} - X_{eh_1})\,dy\right] \tag{139}$$

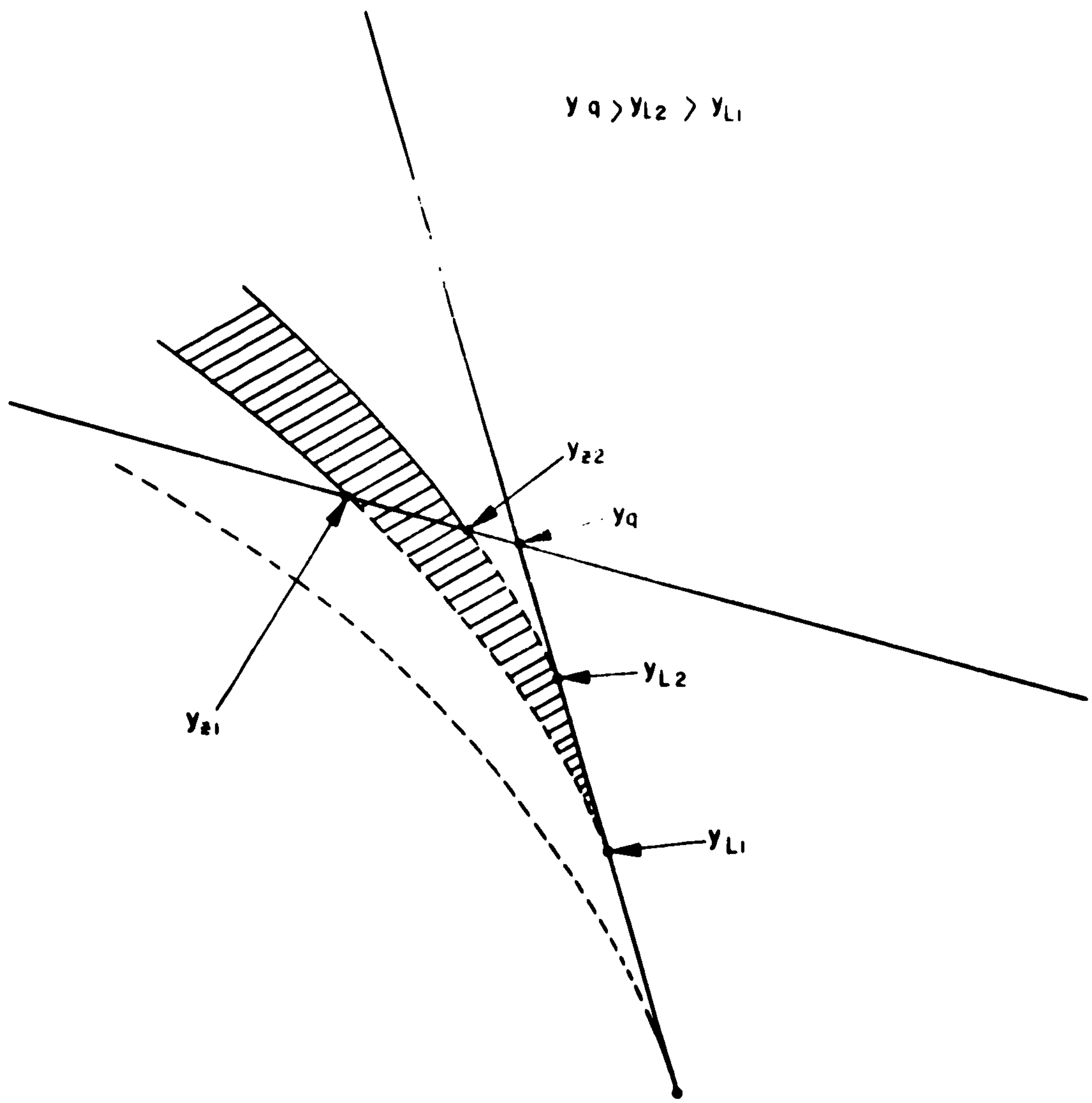

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FIGURE 9

All integration limits, with the exception of y_{z2}, have been defined above.

y_{z2} is defined by equation (137) with a change of subscript.

$$y_{z2} = \frac{-\left(\frac{h_2 \sin\theta}{m_{fc1}}\right) - \left[\frac{a_2'^2}{m_{fc1}^2} - \frac{a_2'^2}{b_2'^2}(h_2 \sin\theta)^2 + \frac{a_2'^4}{b_2'^2}\right]^{\frac{1}{2}}}{\left(\frac{1}{m_{fc1}^2} + \frac{a_2'^2}{b_2'^2}\right)} \tag{140}$$

X_{rc1} and X_{fc1} have been written above (case III c) (2)) in the coordinate system with the origin at the center of the projected ellipse corresponding to h_1.

$$X_{rc1} = \frac{1}{m_{rc1}} y + h_1' \sin\theta \tag{141}$$

$$X_{fc1} = \frac{1}{m_{fc1}} y + h_1 \sin\theta \tag{142}$$

For X_{eh_2} in the above coordinate system, we write

$$X_{eh_2} = \frac{a_2'}{b_2'}(b_2'^2 - y^2)^{\frac{1}{2}} - \gamma \tag{143}$$

where

$$\gamma = (h_2 - h_1) \sin\theta \tag{144}$$

With the above substitutions

$$A' = 2\left\{\int_{y_{L1}}^{y_{L2}}\left[\frac{1}{m_{rc1}}y + h_1'\sin\theta - \frac{a_1'}{b_1'}(b_1'^2 - y^2)^{\frac{1}{2}}\ dy\right]\right.$$

$$+\int_{y_{L2}}^{y_{z2}}\left[\frac{a_2'}{b_2'}(b_2'^2 - y^2)^{\frac{1}{2}} - \gamma - \frac{a_1'}{b_1'}(b_1'^2 - y^2)^{\frac{1}{2}}\ dy\right]$$

$$\left.+\int_{y_{z2}}^{y_{z1}}\left[\frac{1}{m_{fc1}}y + h_1\sin\theta - \frac{a_1'}{b_1'}(b_1'^2 - y^2)^{\frac{1}{2}}\ dy\right]\right\} \tag{145}$$

$$A' = 2\left[\frac{1}{m_{rc1}}\left.\frac{y^2}{2}\right|_{y_{L1}}^{y_{L2}} + (h_1'\sin\theta)\,y\Big|_{y_{L1}}^{y_{L2}} - \frac{a_1'}{b_1'}\int_{y_{L1}}^{y_{L2}}(b_1'^2 - y^2)^{\frac{1}{2}}dy\right.$$

$$+\frac{a_2'}{b_2'}\int_{y_{L2}}^{y_{z2}}(b_2'^2 - y^2)^{\frac{1}{2}}dy - \gamma y\Big|_{y_{L2}}^{y_{z2}} - \frac{a_1'}{b_1'}\int_{y_{L2}}^{y_{z2}}(b_1'^2 - y^2)^{\frac{1}{2}}dy$$

$$\left.+\frac{1}{m_{fc1}}\left.\frac{y^2}{2}\right|_{y_{z2}}^{y_{z1}} + (h_1\sin\theta)\,y\Big|_{y_{z2}}^{y_{z1}} - \frac{a_1'}{b_1'}\int_{y_{z2}}^{y_{z1}}(b_1'^2 - y^2)^{\frac{1}{2}}dy\right] \tag{146}$$

$$
\begin{aligned}
A' = {} & \frac{1}{m_{rc1}} (y_{L2}^2 - y_{L1}^2) + 2(h_1' \sin\theta)(y_{L2} - y_{L1}) \\
& - \frac{a_1'}{b_1'} \left\{ y_{L2}(b_1'^2 - y_{L2}^2)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{L2}}{b_1'}\right) \right. \\
& \left. - \left[y_{L1}(b_1'^2 - y_{L1}^2)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{L1}}{b_1'}\right) \right] \right\} \\
& + \frac{a_2'}{b_2'} \left\{ y_{z2}(b_2'^2 - y_{z2}^2)^{\frac{1}{2}} + b_2'^2 \sin^{-1}\left(\frac{y_{z2}}{b_2'}\right) \right. \\
& \left. - \left[y_{L2}(b_2'^2 - y_{L2}^2)^{\frac{1}{2}} + b_2'^2 \sin^{-1}\left(\frac{y_{L2}}{b_2'}\right) \right] \right\} \\
& - 2\gamma(y_{z2} - y_{L2}) - \frac{a_1'}{b_1'} \left\{ y_{z2}(b_1'^2 - y_{z2}^2)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{z2}}{b_1'}\right) \right. \\
& \left. - \left[y_{L2}(b_1'^2 - y_{L2}^2)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{L2}}{b_1'}\right) \right] \right\} \\
& + \frac{1}{m_{fc1}} (y_{z1}^2 - y_{z2}^2) + 2(h_1 \sin\theta)(y_{z1} - y_{z2}) \\
& - \frac{a_1'}{b_1'} \left\{ y_{z1}(b_1'^2 - y_{z1}^2)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{z1}}{b_1'}\right) \right. \\
& \left. - \left[y_{z2}(b_1'^2 - y_{z2}^2)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{z2}}{b_1'}\right) \right] \right\}
\end{aligned}
\tag{147}
$$

$$A' = \frac{1}{m_{rcl}} (y_{L2}^{2} - y_{L1}^{2}) + 2(h_1' \sin\theta)(y_{L2} - y_{L1})$$

$$+ \frac{1}{m_{fcl}} (y_{z1}^{2} - y_{z2}^{2}) + 2(h_1 \sin\theta)(y_{z1} - y_{z2}) - 2\gamma(y_{z2} - y_{L2})$$

$$+ \frac{a_1'}{b_1'} \left\{ \left[y_{L1}(b_1'^{2} - y_{L1}^{2})^{\frac{1}{2}} + b_1'^{2} \sin^{-1}\left(\frac{y_{L1}}{b_1'}\right) \right] \right.$$

$$\left. - \left[y_{z1}(b_1'^{2} - y_{z1}^{2})^{\frac{1}{2}} + b_1'^{2} \sin^{-1}\left(\frac{y_{z1}}{b_1'}\right) \right] \right\}$$

$$+ \frac{a_2'}{b_2'} \left\{ \left[y_{z2}(b_2'^{2} - y_{z2}^{2})^{\frac{1}{2}} + b_2'^{2} \sin^{-1}\left(\frac{y_{z2}}{b_2'}\right) \right] \right.$$

$$\left. - \left[y_{L2}(b_2'^{2} - y_{L2}^{2})^{\frac{1}{2}} + b_2'^{2} \sin^{-1}\left(\frac{y_{L2}}{b_2'}\right) \right] \right\} \qquad (148)$$

Summarizing case III c) (3), we have

$$A = (h_2'^{2} - h_1'^{2}) \tan\alpha_2 \left(\tan\alpha_2 \cos\theta \left\{ \pi - \sin^{-1}\left[\frac{(\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}}}{\sin\theta} \right] \right\} + (\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}} \right) - A' \text{ (defined by equation (148))}. \qquad (149)$$

$$\alpha_2 < \theta \leq 90^{\circ} \qquad\qquad h_2 > h_1 \geq h_{01} \qquad (150)$$

$$\frac{Q \sin \theta - C_{fcl}}{\frac{1}{m_{fcl}} - \frac{1}{m_{rcl}}} > \frac{b_2'}{C_2'} (C_2'^2 - a_2'^2)^{\frac{1}{2}} > \frac{b_1'}{C_1'} (C_1'^2 - a_1'^2)^{\frac{1}{2}} \quad (151)$$

where:	is defined by equation:
h_1'	(123)
h_2'	(124)
a_1'	(125)
b_1'	(126)
a_2'	(127)
b_2'	(128)
C_1'	(129)
C_2'	(130)
m_{fcl}	(132)
m_{rcl}	(133)
Q	(134)
y_{L1}	(135)
y_{L2}	(84)
y_q	(136)
y_{z1}	(137)
y_{z2}	(140)
γ	(144)
A'	(148)

PART IV SUMMARY OF EQUATIONS

I FRONT CONE $(h_{01} \geqq h_2 > h_1)$ $\alpha_2 > \alpha_1$

a) $0 \leqq \theta \leqq \alpha_1$

$$A = \pi \tan^2 \alpha_1 \cos \theta \,(h_2^{\,2} - h_1^{\,2}) \tag{152}$$

b) $\alpha_1 < \theta \leqq 90^\circ$

$$A = (h_2^{\,2} - h_1^{\,2}) \tan \alpha_1 \left(\tan \alpha_1 \cos \theta \left\{ \pi - \sin^{-1} \left[\frac{(\sin^2 \theta - \tan^2 \alpha_1 \cos^2 \theta)^{\frac{1}{2}}}{\sin \theta} \right] \right\} + (\sin^2 \theta - \tan^2 \alpha_1 \cos^2 \theta)^{\frac{1}{2}} \right) \tag{153}$$

II REAR CONE $(h_2' > h_1' \geqq h_{01})$ $\alpha_2 > \alpha_1$

a) $0 \leqq \theta \leqq \alpha_1$

$$A = \pi \tan^2 \alpha_2 \cos \theta \,(h_2'^{\,2} - h_1'^{\,2}) \tag{154}$$

b) $\alpha_1 < \theta \leqq \alpha_2$

$$(1) \quad \theta \leqq \tan^{-1} \left[\frac{h_1' \tan \alpha_2}{h_1} \right] \tag{155}$$

$$A = \pi \tan^2 \alpha_2 \cos \theta \,(h_2'^{\,2} - h_1'^{\,2}) \tag{156}$$

$$(2)\quad \tan^{-1}\left[\frac{h_1' \tan\alpha_2}{h_1}\right] < \theta < \tan^{-1}\left[\frac{h_2' \tan\alpha_2}{h_2}\right] \tag{157}$$

$$A = \pi \tan^2\alpha_2 \cos\theta\,(h_2'^2 - h_1'^2) - \left(\frac{y_{z1}^2}{m_{fc1}} + 2C_{fc1}y_{z1} - \frac{a_1'}{b_1'}\left\{y_{z1}\left[b_1'^2 - y_{z1}^2\right]^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{z1}}{b_1'}\right)\right\} + 2W_1 y_{z1}\right) \tag{158}$$

$$(3)\quad \tan^{-1}\left[\frac{h_2' \tan\alpha_2}{h_2}\right] \leq \theta \tag{159}$$

$$A = \pi \tan^2\alpha_2 \cos\theta\,(h_2'^2 - h_1'^2) - \left(\left\{\frac{y_{z1}}{m_{fc1}} + 2C_{fc1}y_{z1} - \frac{a_1'}{b_1'}\left[y_{z1}\left(b_1'^2 - y_{z1}^2\right)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{z1}}{b_1'}\right)\right] + 2W_1 y_{z1}\right\} - \left\{\frac{y_{z2}}{m_{fc1}} + 2C_{fc1}y_{z2} - \frac{a_2'}{b_2'}\left[y_{z2}\left(b_2'^2 - y_{z2}^2\right)^{\frac{1}{2}} + b_2'^2 \sin^{-1}\left(\frac{y_{z2}}{b_2'}\right)\right] + 2W_2 y_{z2}\right\}\right) \tag{160}$$

c) $\alpha_2 < \theta \leq 90^\circ$

(1) $y_{L2} > y_{L1} \geq y_q$

$$A = (h_2'^2 - h_1'^2) \tan\alpha_2 \left(\tan\alpha_2 \cos\theta \left\{ \pi - \sin^{-1}\left[\frac{(\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}}}{\sin\theta} \right] \right\} + (\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}} \right) \tag{161}$$

(2) $y_{L2} \geqq y_q > y_{L1}$

$$\begin{aligned} A = {} & (h_2'^2 - h_1'^2) \tan\alpha_2 \left(\tan\alpha_2 \cos\theta \left\{ \pi - \sin^{-1}\left[\frac{(\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}}}{\sin\theta} \right] \right\} \right. \\ & \left. + (\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}} \right) - \left(\frac{y_q^2 - y_{L1}^2}{m_{rc1}} \right. \\ & + (2h_1' \sin\theta)(y_q - y_{L1}) + \frac{a_1'}{b_1'} \left\{ \left[y_{L1} (b_1'^2 - y_{L1}^2)^{\frac{1}{2}} \right. \right. \\ & \left. + b_1'^2 \sin^{-1} \frac{y_{L1}}{b_1'} \right] - \left[y_{z1} (b_1'^2 - y_{z1}^2)^{\frac{1}{2}} \right. \\ & \left. \left. + b_1'^2 \sin^{-1} \frac{y_{z1}}{b_1'} \right] \right\} + \frac{y_{z1}^2 - y_q^2}{m_{fc1}} \\ & \left. + 2(h_1 \sin\theta)(y_{z1} - y_q) \right) \end{aligned} \tag{162}$$

(3) $y_q > y_{L2} > y_{L1}$

$$
\begin{aligned}
A = {} & (h_2'^2 - h_1'^2) \tan\alpha_2 \Bigg(\tan\alpha_2 \cos\theta \\
& \left\{ \pi - \sin^{-1}\left[\frac{(\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}}}{\sin\theta} \right] \right\} \\
& + (\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}} \Bigg) - \Bigg(\frac{1}{m_{rc1}} (y_{L2}^2 - y_{L1}^2) \\
& + 2(h_1' \sin\theta)(y_{L2} - y_{L1}) + \frac{1}{m_{fc1}} (y_{z1}^2 - y_{z2}^2) \\
& + 2(h_1 \sin\theta)(y_{z1} - y_{z2}) - 2\gamma(y_{z2} - y_{L2}) \\
& + \frac{a_1'}{b_1'} \left\{ \left[y_{L1}(b_1'^2 - y_{L1}^2)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{L1}}{b_1'}\right) \right] \right. \\
& \left. - \left[y_{z1}(b_1'^2 - y_{z1}^2)^{\frac{1}{2}} + b_1'^2 \sin^{-1}\left(\frac{y_{z1}}{b_1'}\right) \right] \right\} \\
& + \frac{a_2'}{b_2'} \left\{ \left[y_{z2}(b_2'^2 - y_{z2}^2)^{\frac{1}{2}} + b_2'^2 \sin^{-1}\left(\frac{y_{z2}}{b_2'}\right) \right] \right. \\
& \left. - \left[y_{L2}(b_2'^2 - y_{L2}^2)^{\frac{1}{2}} + b_2'^2 \sin^{-1}\left(\frac{y_{L2}}{b_2'}\right) \right] \right\} \Bigg)
\end{aligned}
\tag{163}
$$

where

$$
h_i' = \left[h_i - h_{01} \tan\alpha_1 \left(\cot\alpha_1 - \cot\alpha_2 \right) \right] \tag{164}
$$

$$a_i' = h_i' \tan\alpha_2 \cos\theta \tag{165}$$

$$b_i' = h_i' \tan\alpha_2 \tag{166}$$

$$C_i' = h_i' \sin\theta \tag{167}$$

$$C_{fcl} = h_{01} \sin\theta \tag{168}$$

$$m_{fcl} = -\frac{\tan\alpha_1}{(\sin^2\theta - \tan^2\alpha_1 \cos^2\theta)^{\frac{1}{2}}} \tag{169}$$

$$m_{rcl} = -\frac{\tan\alpha_2}{(\sin^2\theta - \tan^2\alpha_2 \cos^2\theta)^{\frac{1}{2}}} \tag{170}$$

$$Q = \frac{\tan\alpha_1}{\tan\alpha_2}(h_{01}) \tag{171}$$

$$y_{Li} = \frac{b_i'}{C_i'}(C_i'^2 - a_i'^2)^{\frac{1}{2}} \tag{172}$$

$$y_{zi} = \frac{-\left(\frac{h_i \sin\theta}{m_{fcl}}\right) - \left[\frac{a_i'^2}{m_{fcl}^2} - \frac{a_i'^2}{b_i'^2}(h_i \sin\theta)^2 + \frac{a_i'^4}{b_i'^2}\right]^{\frac{1}{2}}}{\left(\frac{1}{m_{fcl}^2} + \frac{a_i'^2}{b_i'^2}\right)} \tag{173}$$

$$W_i = (h_i - h_{01}) \sin\theta \tag{174}$$

$$\gamma = (h_2 - h_1) \sin\theta \tag{175}$$